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Bright and dark optical solitons in coupled higher-order nonlinear Schrödinger equations through singularity structure analysis

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Abstract. A fairly general form of coupled higher-order nonlinear Schrödinger (CHNLS) equations, which includes the effect of group velocity dispersion (GVD), third-order dispersion, Kerr-law nonlinearity and describing a large class of phenomena involving soliton interactions, has been investigated using Painlevé (P) singularity structure analysis in order to identify the underlying integrable models. The identified integrable models agree well with those obtained from AKNS formulation. In addition, we explicitly obtain the bright and dark N-soliton solutions for the integrable model by using Hirota bilinearization derivable from the P-analysis. The form of the bright one-soliton agrees with the result derivable from the inverse scattering analysis, while that of the remaining higher-order bright solitons and dark N-solitons are reported for the first time, by including the most general linear coupling terms.

1. Introduction

The propagation of optical soliton pulses through a fibre medium is governed by the nonlinear Schrödinger (NLS) family of equations, including their higher-order and coupled versions depending upon the physical situation that is being modelled [1,2]. The NLS equation has two types of soliton solution, namely bright and dark soliton solutions. While the former can exist in the anomalous GVD region [3] where the dispersion and the cubic nonlinear coefficients have identical signs (that is, the product of these two coefficients is greater than zero), the latter can occur in the normal GVD region [4] where those two coefficients take opposite signs (that is, the product of dispersion and cubic nonlinear coefficients is less than zero). Zakharov and Shabat [5] solved exactly the NLS equation by means of the inverse scattering method and noted pulse-like envelope soliton solutions in the anomalous GVD region and, in the case of the normal GVD region, soliton solutions appear in the form of a dip against a uniform background [6]. The former ones are the bright solitons and the latter ones are the dark solitons. Recently the theoretical and experimental aspects of the dark solitons have been reviewed by Kivshar [7].

The role of a set of coupled NLS family of equations becomes quite important to explain the interaction of optical solitons in a two-mode fibre [8–10], birefringent fibre [11–13], directional coupler [14–16], etc. One such a fairly general form of coupled higher-order nonlinear Schrödinger (CHNLS) equations is the generalized version of the higher-order NLS equation [17],

$$
\begin{aligned}
&\mathrm{i}q_{1z} + \mathrm{i}\rho_1 q_{1t} + \frac{\lambda}{2} q_{1tt} + \alpha\lambda(|q_1|^2 + \eta|q_2|^2)q_1 + (\sigma_+ + \sigma_-)q_1 + (\kappa_+ + \mathrm{i}\kappa_-)q_2 \\
&\qquad -\mathrm{i}\varepsilon[q_{1ttt} + \alpha\Omega(|q_1|^2 + \eta|q_2|^2)q_{1t} + \alpha\Omega(q_1^* q_{1t} + \eta q_2^* q_{2t})q_1] = 0 \qquad (1.1a) \\
&\mathrm{i}q_{2z} + \mathrm{i}\rho_2 q_{2t} + \frac{\lambda}{2} q_{2tt} + \alpha\lambda(\eta|q_1|^2 + |q_2|^2)q_2 + (\sigma_+ - \sigma_-)q_2 + (\kappa_+ - \mathrm{i}\kappa_-)q_1 \\
&\qquad -\mathrm{i}\varepsilon[q_{2ttt} + \alpha\Omega(\eta|q_1|^2 + |q_2|^2)q_{2t} + \alpha\Omega(\eta q_1^* q_{1t} + q_2^* q_{2t})q_2] = 0 \qquad (1.1b)
\end{aligned}
$$

where ρ_1, ρ_2, σ_+, σ_-, κ_+, κ_-, η, α, λ, Ω and ε are real parameters, the variables z and t are the normalized distance and time along the fibre respectively and $q_j(z,t)$, $j = 1, 2$ are the normalized envelopes of the two modes. Here the parameter ε approaches the value zero if the pulse width is long compared to the wavelength and the bright and dark soliton solutions of the resultant coupled NLS equations have been recently constructed [18] by deriving the corresponding Hirota bilinear form from the results of Painlevé (P) analysis in the absence of linear cross coupling terms $\kappa_+ = \kappa_- = 0$. The parameter α can take both positive and negative values depending on whether bright or dark solitons are present in the system respectively. For $\alpha = 1$, Tasgal and Potasek [17], using the inverse scattering method, explicitly derived a bright one-soliton solution of the system (1.1) for the parametric restrictions $\Omega = 3$, $\rho_1 = \rho_2$ and $\eta = 1$. They also briefly reviewed the role of the parameters ρ_1, ρ_2, σ_+, σ_-, κ_+, κ_-, ε and η corresponding to the different physical situations that are being modelled from the previous soliton works. It appears that up to this date there has been no systematic work to find bright and dark N-soliton solutions of the CHNLS equations by including the most general linear cross coupling terms κ_+, κ_-.

In this paper, we find explicitly bright and dark N-soliton solutions of the underlying integrable model of the system (1.1) using the relation between P-analysis and Hirota technique. The plan of the paper is as follows. In section 2, in order to identify the underlying integrable models of the system (1.1), we apply P-analysis and note that the identified integrable model, which is the same as obtained by Tasgal and Potasek [17] from AKNS formulation, does not have any restriction on the parameters σ_+, σ_-, κ_+, κ_-, α, λ and ε as such. In section 3, we derive explicitly the bright and dark N-soliton solutions of the integrable model using a Hirota bilinear transformation derivable from the P-analysis. It is also verified in this section that the bright one-soliton obtained here exhibits the same form as that reported from the inverse scattering method [17], and that the remaining higher-order bright and dark solitons are reported here by including the higher-order terms and the most general linear coupling terms systematically, for the first time. Section 4 is devoted to a discussion of the results.

2. Painlevé singularity structure analysis of the CHNLS equations

In recent years, the Painlevé singularity structure analysis has been identified as one of the powerful tools in the search for new integrable systems [19, 20]. The remarkable feature of this analysis, particularly for soliton equations, is that a natural connection exists in relation to the Lax pair, Bäcklund transformation (BT) and Hirota method. Therefore, investigating the underlying integrable models of the coupled soliton equations by means of this analysis is quite interesting [21, 22] and in this section we perform the singular point analysis for the CHNLS equations (1.1).

2.1. Leading order and resonance analysis

In order to apply the P-analysis, we define $q_1 = a$, $q_1^* = b$, $q_2 = c$, $q_2^* = d$ and rewrite (1.1) and its complex conjugate (after a rescaling of q_1 and q_2 by a factor $1/\sqrt{\alpha}$) as

$$\begin{aligned} &\mathrm{i}a_z + \mathrm{i}\rho_1 a_t + \frac{\lambda}{2}a_{tt} + \lambda(ab+\eta cd)a + (\sigma_+ + \sigma_-)a + (\kappa_+ + \mathrm{i}\kappa_-)c \\ &\qquad -\mathrm{i}\varepsilon[a_{ttt} + \Omega(ab+\eta cd)a_t + \Omega(a_t b + \eta c_t d)a] = 0 \end{aligned} \tag{2.1a}$$

$$\begin{aligned} &-\mathrm{i}b_z - \mathrm{i}\rho_1 b_t + \frac{\lambda}{2}b_{tt} + \lambda(ab+\eta cd)b + (\sigma_+ + \sigma_-)b + (\kappa_+ - \mathrm{i}\kappa_-)d \\ &\qquad +\mathrm{i}\varepsilon[b_{ttt} + \Omega(ab+\eta cd)b_t + \Omega(ab_t + \eta cd_t)b] = 0 \end{aligned} \tag{2.1b}$$

$$\begin{aligned} &\mathrm{i}c_z + \mathrm{i}\rho_2 c_t + \frac{\lambda}{2}c_{tt} + \lambda(\eta ab+cd)c + (\sigma_+ - \sigma_-)c + (\kappa_+ - \mathrm{i}\kappa_-)a \\ &\qquad -\mathrm{i}\varepsilon[c_{ttt} + \Omega(\eta ab+cd)c_t + \Omega(\eta a_t b + c_t d)c] = 0 \end{aligned} \tag{2.1c}$$

$$\begin{aligned} &-\mathrm{i}d_z - \mathrm{i}\rho_2 d_t + \frac{\lambda}{2}d_{tt} + \lambda(\eta ab+cd)d + (\sigma_+ + \sigma_-)d + (\kappa_+ + \mathrm{i}\kappa_-)b \\ &\qquad +\mathrm{i}\varepsilon[d_{ttt} + \Omega(\eta ab+cd)d_t + \Omega(\eta ab_t + cd_t)d] = 0. \end{aligned} \tag{2.1d}$$

The singularity structure analysis of (2.1) is carried out by seeking the generalized Laurent expansions in the form

$$a(z,t) = \phi^p \sum_{i=0}^{\infty} a_i(z,t)\phi^i(z,t) \tag{2.2a}$$

$$b(z,t) = \phi^q \sum_{i=0}^{\infty} b_i(z,t)\phi^i(z,t) \tag{2.2b}$$

$$c(z,t) = \phi^r \sum_{i=0}^{\infty} c_i(z,t)\phi^i(z,t) \tag{2.2c}$$

$$d(z,t) = \phi^s \sum_{i=0}^{\infty} d_i(z,t)\phi^i(z,t) \tag{2.2d}$$

in the neighbourhood of the non-characteristic singular manifold $\phi(z,t) = 0$ ($\phi_z, \phi_t \neq 0$) and searching for the conditions under which the solution is free from movable critical manifolds.

Assuming the leading order of the solutions in the form

$$a \simeq a_0\phi^p \qquad b \simeq b_0\phi^q \qquad c \simeq c_0\phi^r \qquad d \simeq d_0\phi^s \tag{2.3}$$

we substitute (2.3) in (2.1) and determine the exponents p, q, r, s and the coefficients a_0, b_0, c_0, d_0 by balancing the dominant terms. In order to simplify the calculations, we make use of the Kruskal ansatz [23] $\phi(z,t) = \psi(z) + t$, where ψ is an arbitrary analytic function of z. Then the coefficient functions a_i, b_i, c_t and d_i in (2.2) will be a function of z alone. It may be noted that the dominant terms are all those terms which are proportional to ε in equations (2.1) and on balancing them, we obtain

$$p+q=-2 \qquad r+s=-2 \tag{2.4a}$$

$$p(p-1)(p-2) + 2\Omega a_0 b_0 p + \Omega\eta c_0 d_0(p+r) = 0 \tag{2.4b}$$

$$q(q-1)(q-2) + 2\Omega a_0 b_0 q + \Omega\eta c_0 d_0(q+s) = 0 \tag{2.4c}$$

$$r(r-1)(r-2) + 2\Omega c_0 d_0 r + \Omega\eta a_0 b_0(p+r) = 0 \tag{2.4d}$$

$$s(s-1)(s-2) + 2\Omega c_0 d_0 s + \Omega\eta a_0 b_0(q+s) = 0. \tag{2.4e}$$

Requiring that the leading order exponents be integers only for the P-property to hold, one easily obtains from equations (2.4) the following three possibilities.

Case I:

$$p = q = r = s = -1 \tag{2.5a}$$

$$a_0 b_0 + \eta c_0 d_0 = \frac{-3}{\Omega} \tag{2.5b}$$

$$\eta a_0 b_0 + c_0 d_0 = \frac{-3}{\Omega} \tag{2.5c}$$

Case II:

$$p = r = -2 \qquad q = s = 0 \tag{2.6a}$$

$$a_0 b_0 + \eta c_0 d_0 = \frac{-6}{\Omega} \tag{2.6b}$$

$$\eta a_0 b_0 + c_0 d_0 = \frac{-6}{\Omega} \tag{2.6c}$$

Case III:

$$p = r = 0 \qquad q = s = -2 \tag{2.7a}$$

$$a_0 b_0 + \eta c_0 d_0 = \frac{-6}{\Omega} \tag{2.7b}$$

$$\eta a_0 b_0 + c_0 d_0 = \frac{-6}{\Omega}. \tag{2.7c}$$

Next, in order to find the resonances, that is the powers at which the arbitrary functions enter into the generalized Laurent expansions (2.2), we expand

$$a = a_0 \phi^p + \cdots + a_j \phi^{p+j} \tag{2.8a}$$

$$b = b_0 \phi^q + \cdots + b_j \phi^{q+j} \tag{2.8b}$$

$$c = c_0 \phi^r + \cdots + c_j \phi^{r+j} \tag{2.8c}$$

$$d = d_0 \phi^s + \cdots + d_j \phi^{s+j} \tag{2.8d}$$

and use them in (2.1). Detailed calculations give the following resonance equations for the exponent j:

(Leading order) case I:

$$j^2(j^2-1)(j-3)(j-4)^2(j-5)[\eta^2(j^4-8j^3+26j^2-40j+33)+2\eta(j^4-8j^3 +20j^2-16j-9)+(j^4-8j^3+14j^2+8j-15)]=0. \tag{2.9}$$

(Leading order) cases II and III:

$$j^2(j+1)(j+2)(j-4)^2(j-5)(j-6)[\eta^2(j^2-5j+6)+\eta(2j^2-10j)+(j^2-5j-6)] \times[\eta^2(j^2-3j+2)+\eta(2j^2-6j-8)+(j^2-3j-10)]=0. \tag{2.10}$$

Considering case I, equation (2.9) gives 12 resonances, out of which eight are integers namely, $-1, 0, 0, 1, 3, 4, 4, 5$ and the remaining four are non-integers/complex in general. However, the roots of the quartic equation for j in the equation (2.9) can become integer for the following three specific values of the parameter η, namely $\eta = 0, 1, 2$. The corresponding integer resonance values are given in table 1.

Similarly considering cases II and III, we find that (2.10) admits eight integer resonances, namely $-2, -1, 0, 0, 4, 4, 5, 6$, besides four non-integer or complex resonances, $\{(5/2) \pm (1/2)[(\eta+49)^{1/2}(\eta+1)^{1/2}]\}$, $\{(3/2) \pm (1/2)[(\eta+49)^{1/2}(\eta+1)^{1/2}]\}$. However, the latter four resonances become integers for the two values $\eta = 0$ and $\eta = 1$ only, while they are again non-integers for the parameter $\eta = 2$. Their explicit values are also given in table 1.

Thus from the resonance analysis, we infer that starting from the three leading order cases I, II and III, and searching for integer resonances in the Laurent expansions (2.8),

Table 1. The leading order behaviour and the resonances for the different (leading order) cases.

η-value	(Leading order) cases	Leading order behaviour p, q, r, s	Leading order behaviour a_0, b_0, c_0, d_0	Resonances (j)
0	I	$p=q=r=s=-1$	$a_0b_0=c_0d_0=-3/\Omega$	$-1,-1,0,0,1,1,3,3,4,4,5,5$
	II	$p=r=-2, q=s=0$	$a_0b_0=c_0d_0=-6/\Omega$	$-2,-2,-1,-1,0,0,4,4,5,5,6,6$
	III	$p=r=0, q=s=-2$		
1	I	$p=q=r=s=-1$	$a_0b_0+c_0d_0=-3/\Omega$	$-1,0,0,0,1,2,2,3,4,4,4,5$
	II	$p=r=-2, q=s=0$	$a_0b_0+c_0d_0=-6/\Omega$	$-2,-1,-1,0,0,0,4,4,4,5,5,6$
	III	$p=r=0, q=s=-2$		
2	I	$p=q=r=s=-1$	$a_0b_0+2c_0d_0=-3/\Omega$, $2a_0b_0+c_0d_0=-3/\Omega$	$-1,0,0,1,1,1,3,3,3,4,4,5$
	II	$p=r=-2, q=s=0$	$a_0b_0+2c_0d_0=-6/\Omega$,	$-2,-1,0,0,4,4,5,6,$
	III	$p=r=0, q=s=-2$	$2a_0b_0+c_0d_0=-6/\Omega$	$(3\pm\sqrt{17})/2, (5\pm\sqrt{17})/2$

only for two parametric choices namely $\eta=0$ and $\eta=1$, all the three cases admit integer resonances, and so the corresponding parametric choices are possible candidates to satisfy the P-property. It is interesting to note that for the parametric choice $\eta=2$, even though the system admits real integer resonances in the (leading order) case I Laurent expansion, it shows the presence of a movable branch point type manifold due to the presence of the resonances $(3\pm\sqrt{17})/2$ and $(5\pm\sqrt{17})/2$ in cases II and III. So this parametric choice does not satisfy the P-property and so the corresponding system is non-integrable.

Now considering the parametric choices $\eta=0$ and $\eta=1$, the resonances are seen from table 1 to be as follows.

$\eta=0$:

$$\text{Case I:} \quad j=-1,-1,0,0,1,1,3,3,4,4,5,5 \tag{2.11}$$

$$\text{Cases II/III:} \quad j=-2,-2,-1,-1,0,0,4,4,5,5,6,6 \tag{2.12}$$

$\eta=1$:

$$\text{Case I:} \quad j=-1,0,0,0,1,2,2,3,4,4,4,5 \tag{2.13}$$

$$\text{Cases II/III:} \quad j=-2,-1,-1,0,0,0,4,4,4,5,5,6. \tag{2.14}$$

In order that the P-property is satisfied for these cases, we have to now ensure that sufficient number of arbitrary functions exist at the appropriate resonance values in all the cases I and II/III separately for the parametric choices $\eta=1$ and $\eta=0$.

2.2. *Arbitrary functions for the parametric choice $\eta=1$*

Considering the leading order case I for the parametric choice $\eta=1$, we now discuss briefly the search for arbitrary functions at the resonance values $j=-1,0,0,0,1,2,2,3,4,4,4,5$. Obviously $j=-1$ corresponds to the arbitrariness of the singular manifold. From the leading order results of equations (2.5*b*, *c*) it is also clear that for $\eta=1$ the four functions a_0, b_0, c_0, d_0 are connected by the only relation

$$a_0b_0+c_0d_0=(-3/\Omega) \tag{2.15}$$

ensuring that three of them are arbitrary. This agrees with the resonance values $j=0,0,0$.

Proceeding further, we substitute the Laurent series solutions (2.2) in (2.1) and collect the coefficients of different powers of ϕ, so that we can evaluate the further coefficients.

By collecting the coefficients of $(\phi^{-3}, \phi^{-3}, \phi^{-3}, \phi^{-3})$, we get

$$\begin{bmatrix} a_0b_0 - \frac{3}{\Omega} & 2a_0^2 & a_0d_0 & 2a_0c_0 \\ 2b_0^2 & a_0b_0 - \frac{3}{\Omega} & 2b_0d_0 & b_0c_0 \\ b_0c_0 & 2a_0c_0 & c_0d_0 - \frac{3}{\Omega} & 2c_0^2 \\ 2b_0d_0 & a_0d_0 & 2d_0^2 & c_0d_0 - \frac{3}{\Omega} \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \frac{\mathrm{i}\lambda}{\varepsilon\Omega}\left(1 - \frac{3}{\Omega}\right) \begin{bmatrix} a_0 \\ b_0 \\ c_0 \\ d_0 \end{bmatrix}. \tag{2.16}$$

Using (2.15) in (2.16), we obtain

$$c_0a_1 - a_0c_1 = b_0a_1 + a_0b_1 = d_0b_1 - b_0d_1 = 0 \tag{2.17}$$

along with the additional parametric restriction

$$\lambda = 0 \qquad \text{or} \qquad \Omega = 3. \tag{2.18}$$

From (2.17) it is clear that one of the four functions a_1, b_1, c_1, d_1 is arbitrary provided the additional parametric restriction (2.18) is also satisfied in addition to $\eta = 1$, which agrees with the resonance value $j = 1$. Similarly from the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2})$, we see that among the four functions a_2, b_2, c_2, d_2 two are arbitrary for both $\lambda = 0$ or $\Omega = 3$ and $\rho_1 = \rho_2 = \rho$, which is in agreement with the resonance values $j = 2, 2$. In this way by proceeding further and collecting the coefficients of $(\phi^{-1}, \phi^{-1}, \phi^{-1}, \phi^{-1})$, $(\phi^0, \phi^0, \phi^0, \phi^0)$ and $(\phi^1, \phi^1, \phi^1, \phi^1)$, we establish the required number of arbitrary functions corresponding to the resonance values $j = 3$, $j = 4, 4, 4$ and $j = 5$ without any additional restrictions on the parameters.

We thus find that the (leading order) case I Laurent expansion for $\eta = 1$ admits 12 arbitrary functions without the introduction of any movable critical manifold for the specific parametric choices $\lambda = 0$ or $\Omega = 3$ and $\rho_1 = \rho_2$.

One can proceed systematically in an altogether analogous way for the other type of Laurent expansions corresponding to the (leading order) cases II and III and establish that the corresponding Laurent series are also free from movable critical singularity manifolds, except that there are only 10 arbitrary functions (due to the presence of resonance values -2 and double -1's) in these cases.

Thus we conclude that in all the three cases I–III the Laurent expansions are free from movable critical manifolds, in which the case I expansions contain the full compliment of 12 arbitrary functions corresponding to the four coupled third-order partial differential equations (2.1) for the parametric restrictions

$$\text{(i)}\ \eta = 1 \qquad \Omega = 3 \qquad \rho_1 = \rho_2 \tag{2.19a}$$

$$\text{(ii)}\ \eta = 1 \qquad \lambda = 0 \qquad \rho_1 = \rho_2. \tag{2.19b}$$

Hence (1.1) with the above two parametric choices satisfy the P-property.

2.3. *Arbitrary functions for the parametric choice* $\eta = 0$

From (2.11), for $\eta = 0$ we have the resonance values $j = -1, -1, 0, 0, 1, 1, 3, 3, 4, 4, 5, 5$ in the (leading order) case I, with leading order coefficients as noted in table 1. We again look for the existence of the Laurent expansion about the singularity manifold as in the previous case, step by step. The detailed analysis shows that in order that no movable critical manifold is present in the Laurent expansion, one of the following additional constraints on the parameters should hold:

$$\text{(i)}\ \Omega = 3 \qquad \kappa_+ = \kappa_- = 0\ (\text{besides}\ \eta = 0) \tag{2.20a}$$

$$\text{(ii)}\ \lambda = 0 \qquad \kappa_+ = \kappa_- = 0\ (\text{besides}\ \eta = 0). \tag{2.20b}$$

Both the above conditions, however, imply that (1.1) becomes decoupled into two independent Hirota equations [24] (or their special cases), each of which is obviously integrable. This is in confirmity with the presence of double -1 values in the resonances too.

One can also check that in the leading order cases II and III, again no movable critical manifold is introduced for the above choices (2.19), (2.20). Thus for the above choice of parameters the equations (1.1) also satisfy the P-property.

2.4. Results of the P-analysis

Combining all the above facts, we can now conclude that the system (1.1) possesses the P-property only for four sets of parametric restrictions, among which two appear from the parametric choice $\eta = 1$, namely (i) $\Omega = 3$ and $\rho_1 = \rho_2$ and (ii) $\lambda = 0$ and $\rho_1 = \rho_2$, and the remaining two are from the parametric choice $\eta = 0$, that is, (i) $\Omega = 3$ and $\kappa_+ = \kappa_- = 0$ and (ii) $\lambda = 0$ and $\kappa_+ = \kappa_- = 0$. The latter two cases, as noted above, correspond to Hirota equations only and so no further analysis is needed for them.

The equations corresponding to the parametric restrictions $\Omega = 3$ and $\rho_1 = \rho_2 = \rho$ (say) in the $\eta = 1$ case can be written from (1.1) as

$$\begin{aligned} &\mathrm{i}q_{1z} + \mathrm{i}\rho q_{1t} + \frac{\lambda}{2}q_{1tt} + \alpha\lambda(|q_1|^2 + |q_2|^2)q_1 + (\sigma_+ + \sigma_-)q_1 + (\kappa_+ + \mathrm{i}\kappa_-)q_2 \\ &\qquad -\mathrm{i}\varepsilon[q_{1ttt} + 3\alpha(|q_1|^2 + |q_2|^2)q_{1t} + 3\alpha(q_1^*q_{1t} + q_2^*q_{2t})q_1] = 0 \end{aligned} \tag{2.21a}$$

$$\begin{aligned} &\mathrm{i}q_{2z} + \mathrm{i}\rho q_{2t} + \frac{\lambda}{2}q_{2tt} + \alpha\lambda(|q_1|^2 + |q_2|^2)q_2 + (\sigma_+ - \sigma_-)q_2 + (\kappa_+ - \mathrm{i}\kappa_-)q_1 \\ &\qquad -\mathrm{i}\varepsilon[q_{2ttt} + 3\alpha(|q_1|^2 + |q_2|^2)q_{2t} + 3\alpha(q_1^*q_{1t} + q_2^*q_{2t})q_2] = 0. \end{aligned} \tag{2.21b}$$

One can easily check that the form of the equation corresponding to the second P-case with the parametric choice $\lambda = 0$, $\rho_1 = \rho_2$ and $\eta = 1$ is again the same as (2.21) with the restriction $\lambda = 0$ and a rescaling of q_1 and q_2. The integrable model (2.21) includes both the most general linear coupling terms $\kappa_+, \kappa_-, \sigma_+, \sigma_-$ and the higher-order terms systematically and its further study is of considerable importance. We also note at this point that a partial but incomplete P-analysis has been performed very recently [25] for a system similar to the above integrable equations (2.21), but excluding the linear cross coupling terms κ_+, κ_- and self-coupling terms σ_+, σ_- and without finding the soliton solutions.

Now, (2.21) can be rewritten in a simplified form by making the linear transformation [17]

$$q_1 = \exp(\mathrm{i}\Theta/2)[q_1' \cos(\theta/2) - q_2' \sin(\theta/2)] \tag{2.22a}$$

$$q_2 = \exp(-\mathrm{i}\Theta/2)[q_2' \cos(\theta/2) + q_1' \sin(\theta/2)] \tag{2.22b}$$

where

$$\tan\Theta = \frac{\kappa_-}{\kappa_+} \qquad \tan\theta = \frac{1}{\sigma_-}[(\kappa_+^2 + \kappa_-^2)^{1/2}] \tag{2.22c}$$

and subsequently a further transformation

$$q_1' = \exp[\mathrm{i}(\sigma_+ + \sigma_-')z]q_1'' \qquad q_2' = \exp[\mathrm{i}(\sigma_+ - \sigma_-')z]q_2'' \tag{2.23}$$

where

$$\sigma_-' = \sigma_-^2 + \kappa_+^2 + \kappa_-^2 \tag{2.24}$$

as (after dropping the primes)

$$\begin{aligned} &\mathrm{i}q_{1z} + \frac{\lambda}{2}q_{1tt} + \alpha\lambda(|q_1|^2 + |q_2|^2)q_1 - \mathrm{i}\varepsilon[q_{1ttt} + 3\alpha(|q_1|^2 + |q_2|^2)q_{1t} \\ &\qquad +3\alpha(q_1^*q_{1t} + q_2^*q_{2t})q_1] = 0 \end{aligned} \tag{2.25a}$$

$$\mathrm{i}q_{2z} + \frac{\lambda}{2}q_{2tt} + \alpha\lambda(|q_1|^2 + |q_2|^2)q_2 - \mathrm{i}\varepsilon[q_{2ttt} + 3\alpha(|q_1|^2 + |q_2|^2)q_{2t} + 3\alpha(q_1^* q_{1t} + q_2^* q_{2t})q_2] = 0. \tag{2.25b}$$

In the above, the terms proportional to ρ have been removed without affecting the other terms by introducing the new variables $t' = t - \rho z$, and $z' = z$. For $\varepsilon = 0$, the above system (2.25) reduces to the well known integrable model proposed by Manakov [26]. Recently Kaup and Malomed [27] have pointed out that the Manakov model covers, besides the birefringence property, many other physical phenomena such as soliton trapping and daughter wave ('shadow') formation in optical fibres. The terms proportional to ε turn out to be important [28, 29] as perturbation terms in order to govern some nonlinear short-pulse propagation and in such a situation the integrable model (2.25) can assume greater physical significance as the integrable generalization of Manakov model. Further when one neglects fibre loss [30, 31], our model (2.25) can also describe the propagation in the femtosecond region. In the next section, we will study the soliton solutions of the system (2.25).

3. Hirota's bilinearlization and soliton solutions of the integrable CHNLS equations

One of the interesting aspects of the P-analysis [18–22] as carried out in the last section is that the singular expansion obtained for the solution of the partial differential equation can be used to construct the BT and Hirota bilinear form. Here we will consider the system (2.25) for this purpose. Now by truncating the Laurent expansion (2.2) with (2.5a), up to the constant-level term, and noting that the transformations (2.22)–(2.23) will not affect the qualitative form of the Laurent expansion for (2.25), we can formally write the BT as

$$q_1 = a = a_0\phi^{-1} + a_1 \qquad q_1^* = b = b_0\phi^{-1} + b_1$$
$$q_2 = c = c_0\phi^{-1} + c_1 \qquad q_2^* = d = d_0\phi^{-1} + d_1 \tag{3.1}$$

where (a, b, c, d) and (a_1, b_1, c_1, d_1) satisfy (2.25). In order to derive the Hirota bilinear form, we consider the vacuum solutions $a_1 = b_1 = c_1 = d_1 = 0$ in (3.1). Then we have

$$q_1 = a_0\phi^{-1} \qquad q_1^* = b_0\phi^{-1} \qquad q_2 = c_0\phi^{-1} \qquad q_2^* = d_0\phi^{-1}. \tag{3.2}$$

This suggests us to take the Hirota bilinear transformation in the form

$$q_1 = \frac{g}{f} \qquad q_2 = \frac{h}{f} \tag{3.3}$$

where $g(z, t)$, $h(z, t)$ are complex functions and $f(z, t)$ is a real function.

Using (3.3) and the Hirota bilinear operators

$$D_t^m D_z^n (g \cdot f) = (\partial_t - \partial_{t'})^m (\partial_z - \partial_{z'})^n g(z, t) f(z', t')|_{z'=z, t'=t} \tag{3.4}$$

equations (2.25) can be rewritten as

$$f^2[(\mathrm{i}D_z + \tfrac{1}{2}\lambda D_t^2 - \mathrm{i}\varepsilon D_t^3)g \cdot f] + 3\alpha\mathrm{i}\varepsilon f h^* D_t g \cdot h - [D_t^2 f \cdot f - 2\alpha(gg^* + hh^*)] \times[(\tfrac{1}{2}\lambda - 3\mathrm{i}\varepsilon D_t)g \cdot f] = 0 \tag{3.5a}$$

$$f^2[(\mathrm{i}D_z + \tfrac{1}{2}\lambda D_t^2 - \mathrm{i}\varepsilon D_t^3)h \cdot f] - 3\alpha\mathrm{i}\varepsilon f g^* D_t g \cdot h - [D_t^2 f \cdot f - 2\alpha(gg^* + hh^*)] \times[(\tfrac{1}{2}\lambda - 3\mathrm{i}\varepsilon D_t)h \cdot f] = 0. \tag{3.5b}$$

3.1. Bright solitons

Equations (3.5) can be decoupled as

$$\mathcal{A}_1 g \cdot f = 0 \qquad \mathcal{A}_1 h \cdot f = 0 \qquad \mathcal{A}_2 f \cdot f = gg^* + hh^* \qquad \mathcal{A}_3 g \cdot h = 0 \tag{3.6}$$

where the bilinear operators $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are defined as

$$\mathcal{A}_1 = \left(\mathrm{i}D_z + \frac{\lambda}{2}D_t^2 - \mathrm{i}\varepsilon D_t^3\right) \qquad \mathcal{A}_2 = \frac{1}{2\alpha}D_t^2 \qquad \mathcal{A}_3 = D_t. \tag{3.7}$$

For finding the soliton solutions, we proceed in the standard way [24, 32]. For example, in order to find the one-soliton solution, we assume

$$g = \chi g_1 \qquad h = \chi h_1 \qquad f = 1 + \chi^2 f_2 \tag{3.8}$$

where χ is an arbitrary parameter. Substituting (3.8) into (3.6) and then collecting the terms of similar powers in χ, we obtain

$$\chi: \quad \mathcal{A}_1 g_1 \cdot 1 = 0 \qquad \mathcal{A}_1 h_1 \cdot 1 = 0 \tag{3.9}$$

$$\chi^2: \quad \mathcal{A}_2(1 \cdot f_2 + f_2 \cdot 1) = (g_1 g_1^* + h_1 h_1^*) = 0 \qquad \mathcal{A}_3 g_1 \cdot h_1 = 0 \tag{3.10}$$

$$\chi^3: \quad \mathcal{A}_1 g_1 \cdot f_2 = 0 \qquad \mathcal{A}_1 h_1 \cdot f_2 = 0 \tag{3.11}$$

$$\chi^4: \quad \mathcal{A}_2 f_2 \cdot f_2 = 0. \tag{3.12}$$

One can easily check that the solution which is consistent with the system (3.9)–(3.12) is

$$g_1 = \exp(\eta_1) \qquad h_1 = \exp(\eta_1 + \varepsilon_0) \qquad f_2 = \frac{\alpha(1 + \exp(\varepsilon_0 + \varepsilon_0^*))}{(l_1 + l_1^*)^2}\exp(\eta_1 + \eta_1^*) \tag{3.13}$$

where

$$\eta_1 = l_1[t + l_1(\mathrm{i}\tfrac{1}{2}\lambda + \varepsilon l_1)z] + \eta_1^{(0)} \tag{3.14}$$

and in which l_1, $\eta_1^{(0)}$ and ε_0 are all complex constants in general and the symbol $*$ indicates complex conjugate. Using (3.13) into (3.8) and then in (3.3), after absorbing χ, the bright one-soliton solution can be easily worked out to be

$$q_1 = \frac{\varepsilon_1 l_{1\mathrm{R}} \exp\{\mathrm{i}[l_{1\mathrm{I}}t - (\frac{1}{2}\lambda(l_{1\mathrm{I}}^2 - l_{1\mathrm{R}}^2) + \varepsilon l_{1\mathrm{I}}(l_{1\mathrm{I}}^2 - 3l_{1\mathrm{R}}^2))z + \eta_{1\mathrm{I}}^{(0)}]\}}{\cosh\{l_{1\mathrm{R}}t - l_{1\mathrm{R}}[\lambda l_{1\mathrm{I}} + \varepsilon(3l_{1\mathrm{I}}^2 - l_{1\mathrm{R}}^2)]z + \delta^{(0)}\}} \tag{3.15a}$$

$$q_2 = \frac{\varepsilon_2 l_{1\mathrm{R}} \exp\{\mathrm{i}[l_{1\mathrm{I}}t - (\frac{1}{2}\lambda(l_{1\mathrm{I}}^2 - l_{1\mathrm{R}}^2) + \varepsilon l_{1\mathrm{I}}(l_{1\mathrm{I}}^2 - 3l_{1\mathrm{R}}^2))z + \eta_{1\mathrm{I}}^{(0)}]\}}{\cosh\{l_{1\mathrm{R}}t - l_{1\mathrm{R}}[\lambda l_{1\mathrm{I}} + \varepsilon(3l_{1\mathrm{I}}^2 - l_{1\mathrm{R}}^2)]z + \delta^{(0)}\}} \tag{3.15b}$$

where

$$\varepsilon_1 = \left[\frac{1}{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}\right]^{1/2} \qquad \varepsilon_2 = \left[\frac{\exp(2\varepsilon_0)}{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}\right]^{1/2} \tag{3.16}$$

$$\delta^{(0)} = \eta_{1\mathrm{R}}^{(0)} + \frac{1}{2}\ln\left[\frac{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}{(l_1 + l_1^*)^2}\right] \tag{3.17}$$

and from (3.16), we also have

$$|\varepsilon_1|^2 + |\varepsilon_2|^2 = \frac{1}{\alpha}. \tag{3.18}$$

In the above the subscripts R and I denote the real and imaginary part. It is obvious from (3.17) and (3.18) that α must be a positive real constant, which naturally corresponds to an anomalous region admitting bright solitons. Further the evolution of the intensity profile of q_1 is shown in figure 1(*a*) for the parametric choices $|\varepsilon_1|^2 = 0.3$, $l_{1\mathrm{R}} = 1$, $l_{1\mathrm{I}} = 2$, $\lambda = 1$ and $\varepsilon = 0.05$ (similar form can be drawn for q_2 also). The solution (3.15) in which ε plays a considerable role on the velocity of the soliton reduces to the case of the Manakov model while $\varepsilon = 0$ and the corresponding changes are obvious from figure 1(*b*). After substituting, the bright one-soliton solution (3.15) in the transformations (2.22) and (2.23), we note that the resultant solution is in agreement with the result reported from the inverse scattering method by Tasgal and Potasek [17].

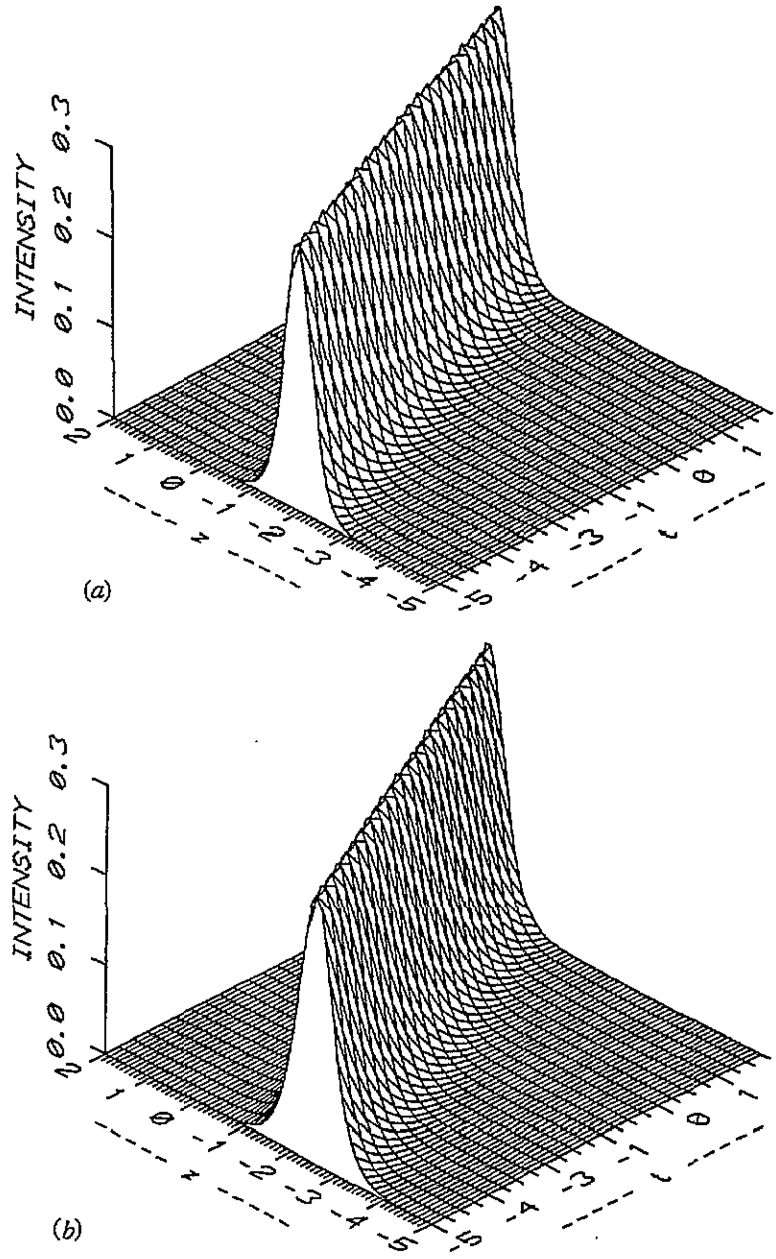

Figure 1. (*a*) Bright one-soliton intensity profile $|q_1|^2$ against z and t in the anomalous GVD region with $\varepsilon = 0.05$ of the integrable model (2.25). (*b*) Same as in (*a*) with $\varepsilon = 0$. Note the broadening of soliton width.

Next in order to find the bright two-soliton solutions, we can assume

$$g = \chi g_1 + \chi^3 g_3 \qquad h = \chi h_1 + \chi^3 h_3 \qquad f = 1 + \chi^2 f_2 + \chi^4 f_4 \quad (3.19)$$

and proceed as in the case of the one-soliton solution; we obtain a system of bilinear equations. On solving them consistently, we obtain

$$g_1 = \exp(\eta_1) + \exp(\eta_2) \quad (3.20a)$$

$$h_1 = \exp(\eta_1 + \varepsilon_0) + \exp(\eta_2 + \varepsilon_0) \tag{3.20b}$$

$$f_2 = a(1, 1^*)\exp(\eta_1 + \eta_1^*) + a(1, 2^*)\exp(\eta_1 + \eta_2^*) + a(2, 1^*)\exp(\eta_2 + \eta_1^*) + a(2, 2^*)\exp(\eta_2 + \eta_2^*) \tag{3.20c}$$

$$g_3 = a(1, 2, 1^*)\exp(\eta_1 + \eta_2 + \eta_1^*) + a(1, 2, 2^*)\exp(\eta_1 + \eta_2 + \eta_2^*) \tag{3.20d}$$

$$h_3 = a(1, 2, 1^*)\exp(\eta_1 + \eta_2 + \eta_1^* + \varepsilon_0) + a(1, 2, 2^*)\exp(\eta_1 + \eta_2 + \eta_2^* + \varepsilon_0) \tag{3.20e}$$

$$f_4 = a(1, 2, 1^*, 2^*)\exp(\eta_1 + \eta_2 + \eta_1^* + \eta_2^*) \tag{3.20f}$$

where

$$\eta_j = l_j[t + l_j(\mathrm{i}\tfrac{1}{2}\lambda + \varepsilon l_j)z] + \eta_j^{(0)} \qquad j = 1, 2 \tag{3.21}$$

$$a(i, j^*) = \frac{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}{(l_i + l_j^*)^2} \qquad a(i, j) = \frac{(l_i - l_j)^2}{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}$$

$$a(i^*, j^*) = \frac{(l_i^* - l_j^*)^2}{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]} \tag{3.22}$$

$$a(i, j, k^*) = a(i, j)a(i, k^*)a(j, k^*) \tag{3.23}$$

and

$$a(i, j, k^*, l^*) = a(i, j)a(i, k^*)a(i, l^*)a(j, k^*)a(j, l^*)a(k^*, l^*). \tag{3.24}$$

Here l_j, $\eta_j^{(0)}$ and ε_0 are all complex constants. Using (3.20)–(3.24) into (3.19) and then in (3.3), the two-soliton solutions of (2.25) are obtained explicitly.

In this way, proceeding further one can generalize the expression for g, h and f corresponding to the N-soliton solutions as

$$g = \sum_{\mu=0,1} M_1(\mu)\exp\left(\sum_{j=1}^{2N}\mu_j\eta_j + \sum_{1\leqslant i<j}^{2N}\mu_i\mu_j\phi_{ij}\right) \tag{3.25}$$

$$h = \sum_{\mu=0,1} M_2(\mu)\exp\left(\sum_{j=1}^{2N}\mu_j\eta_j + \sum_{1\leqslant i<j}^{2N}\mu_i\mu_j\phi_{ij}\right) \tag{3.26}$$

$$f = \sum_{\mu=0,1} M_3(\mu)\exp\left(\sum_{j=1}^{2N}\mu_j\eta_j + \sum_{1\leqslant i<j}^{2N}\mu_i\mu_j\phi_{ij}\right) \tag{3.27}$$

where

$$\eta_j = l_j[t + l_j(\mathrm{i}\tfrac{1}{2}\lambda + \varepsilon l_j)z] + \eta_j^{(0)} \qquad j = 1, 2, \ldots, 2N \tag{3.28}$$

$$\eta_{j+N} = \eta_j^* \qquad l_{j+N} = l_j^* \qquad \text{for } j = 1, 2, \ldots, N \tag{3.29}$$

$$\exp(\phi_{ij}) = \frac{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]}{(l_i + l_j)^2} \qquad \text{for } i = 1, 2, \ldots, N \text{ and } j = N+1, \ldots, 2N \tag{3.30}$$

$$\exp(\phi_{ij}) = \frac{(l_i - l_j)^2}{\alpha[1 + \exp(\varepsilon_0 + \varepsilon_0^*)]} \qquad \text{for } i = 1, 2, \ldots, N \text{ and } j = 1, 2, \ldots, N; \quad i = N+1, \ldots, 2N \text{ and } j = N+1, \ldots, 2N \tag{3.31}$$

and

$$M_1(\mu) = \begin{cases} 1 & \text{when } 1 + \sum_{i=1}^{N}\mu_{i+N} = \sum_{i=1}^{N}\mu_i \\ 0 & \text{otherwise} \end{cases} \tag{3.32}$$

$$M_2(\mu) = \begin{cases} e^{\varepsilon_0} & \text{when } 1 + \sum_{i=1}^{N}\mu_{i+N} = \sum_{i=1}^{N}\mu_i \\ 0 & \text{otherwise} \end{cases} \tag{3.33}$$

$$M_3(\mu) = \begin{cases} 1 & \text{when } \sum_{i=1}^{N} \mu_{i+N} = \sum_{i=1}^{N} \mu_i \\ 0 & \text{otherwise.} \end{cases} \qquad (3.34)$$

3.2. Dark solitons

Now in order to find the dark solitons, (3.5) can be decoupled into the set of bilinear equations as

$$\mathcal{B}_1 g \cdot f = 0 \qquad \mathcal{B}_1 h \cdot f = 0 \qquad \mathcal{B}_2 f \cdot f = gg^* + hh^* \qquad \mathcal{A}_3 g \cdot h = 0 \qquad (3.35)$$

where the bilinear operators $\mathcal{B}_1$ and $\mathcal{B}_2$ are defined as

$$\mathcal{B}_1 = \left(\mathrm{i}D_z + \frac{\lambda}{2}D_t^2 - \mathrm{i}\varepsilon D_t^3 - 3\mathrm{i}\varepsilon\Gamma D_t + \frac{\lambda}{2}\Gamma\right) \qquad \mathcal{B}_2 = \frac{1}{2\alpha}(D_t^2 + \Gamma) \qquad (3.36)$$

in which Γ is a constant to be determined.

For constructing the dark soliton solutions, we assume

$$g = g_0(1 + \chi g_1 + \chi^2 g_2 + \cdots) \qquad h = h_0(1 + \chi h_1 + \chi^2 h_2 + \cdots)$$
$$f = 1 + \chi f_1 + \chi^2 f_2 + \cdots. \qquad (3.37)$$

Substituting (3.37) into (3.35) and then collecting the coefficients of $\chi^{(0)}$, we get

$$\mathcal{B}_1 g_0 \cdot 1 = 0 \qquad \mathcal{B}_1 h_0 \cdot 1 = 0 \qquad g_0 g_0^* + h_0 h_0^* = \frac{\Gamma}{2\alpha} \qquad \mathcal{A}_3 g_0 \cdot h_0 = 0. \qquad (3.38)$$

A set of solutions to (3.38) can easily be written as

$$g_0 = \tau_1 \exp(\mathrm{i}\zeta_1) \qquad h_0 = \tau_2 \exp(\mathrm{i}\zeta_1) \qquad (3.39)$$

where

$$\zeta_1 = K_1 t - [-\tfrac{1}{2}\Gamma\lambda + K_1(\tfrac{1}{2}\lambda K_1 + \varepsilon K_1^2 - 3\varepsilon\Gamma)]z + \zeta_1^{(0)} \qquad (3.40)$$

and τ_1 and τ_2 are connected by the relation

$$\tau_1^2 + \tau_2^2 = \frac{\Gamma}{2\alpha} \qquad (3.41)$$

in which K_1, $\zeta_1^{(0)}$ and (τ_1, τ_2) are real constants.

Using (3.37), (3.39) and the usual Hirota identities [32], equations (3.35) can be rewritten as

$$\mathcal{C}_1 G \cdot f = 0 \qquad \mathcal{C}_1 H \cdot f = 0 \qquad \mathcal{B}_2 f \cdot f = \tau_1^2 GG^* + \tau_2^2 HH^*$$
$$\mathcal{A}_3 G \cdot H = 0 \qquad (3.42)$$

where

$$\mathcal{C}_1 = \{\mathrm{i}D_z + \mathrm{i}[3\varepsilon(K_1^2 - \Gamma) + K_1\lambda]D_t + (3\varepsilon K_1 + \tfrac{1}{2}\lambda)D_t^2 - \mathrm{i}\varepsilon D_t^3\}$$
$$G = (1 + \chi g_1 + \chi^2 g_2 + \cdots) \qquad H = (1 + \chi h_1 + \chi^2 h_2 + \cdots) = 0. \qquad (3.43)$$

Now for obtaining the dark one-soliton solution, we set $g_j = h_j = f_j = 0$ for $j \geqslant 2$ and then collect the terms with the same power in χ. Then we have

χ^1:

$$\mathcal{C}_1(1 \cdot f_1 + g_1 \cdot 1) = 0 \qquad \mathcal{C}_1(1 \cdot f_1 + h_1 \cdot 1) = 0$$
$$\mathcal{B}_2(1 \cdot f_1 + f_1 \cdot 1) = \tau_1^2(g_1 + g_1^*) + \tau_2^2(h_1 + h_1^*)$$
$$\mathcal{A}_3(1 \cdot h_1 + g_1 \cdot 1) = 0 \qquad (3.44)$$

χ^2:

$$\mathcal{C}_1 g_1 \cdot f_1 = 0 \qquad \mathcal{C}_1 h_1 \cdot f_1 = 0 \qquad \mathcal{B}_2 f_1 \cdot f_1 = \tau_1^2 g_1 g_1^* + \tau_2^2 h_1 h_1^*$$
$$\mathcal{A}_3 g_1 \cdot h_1 = 0. \qquad (3.45)$$

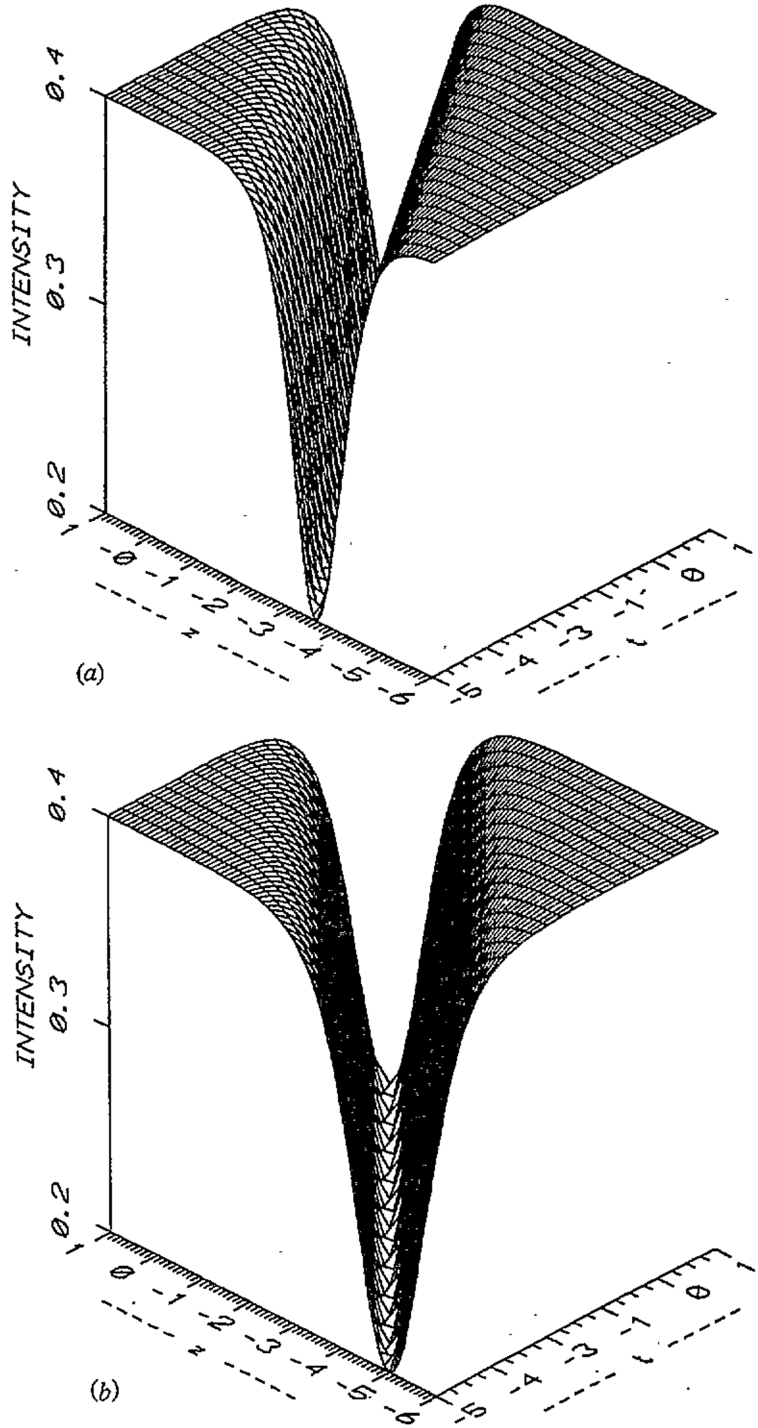

Figure 2. (a) Dark one-soliton intensity profile $|q_1|^2$ against z and t in the normal GVD region with $\varepsilon = 0.05$ in (2.25). (b) Same as in (a) with $\varepsilon = 0$.

One can easily check that the systems (3.44), (3.45) admit the following solutions.

$$g_1 = h_1 = z_1 \exp(\xi_1) \qquad f_1 = \exp(\xi_1) \tag{3.46}$$

where

$$\xi_1 = P_1 t - \{[3\varepsilon(K_1^2 - \Gamma) + K_1\lambda]P_1 - \varepsilon P_1^3 - P_1(3\varepsilon K_1 + \tfrac{1}{2}\lambda)[4\alpha_1(\tau_1^2 + \tau_2^2) - P_1^2]^{1/2}\}z + \xi_1^{(0)} \tag{3.47}$$

and

$$Z_1 = \frac{-P_1 + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_1^2]^{1/2}}{P_1 + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_1^2]^{1/2}} \tag{3.48}$$

in which P_1 and $\xi_1^{(0)}$ are real constants, and the parameter α_1 is taken as $\alpha = -\alpha_1$. The expression (3.48) for the complex constant Z_1 shows that $|Z_1|^2 = 1$. Since here we have assumed f as real, ξ_1 must be real. This assumption is valid only if $\alpha_1 = -\alpha$ in (3.47) is greater than zero (that is $\alpha < 0$) such that $4\alpha_1(\tau_1^2 + \tau_2^2) > P_1^2$ and which naturally corresponds to the normal GVD region where dark solitons appear. Now using (3.39) and (3.46) in (3.37) and then in (3.3), after absorbing χ, the dark one-soliton solution can be derived as

$$q_1 = \tfrac{1}{2}\tau_1 \exp(\mathrm{i}\zeta_1)[(1 + Z_1) - (1 - Z_1)\tanh(\xi_1/2)] \tag{3.49a}$$

$$q_2 = \tfrac{1}{2}\tau_2 \exp(\mathrm{i}\zeta_1)[(1 + Z_1) - (1 - Z_1)\tanh(\xi_1/2)] \tag{3.49b}$$

where

$$\tau_1^2 + \tau_2^2 = \frac{\Gamma}{2\alpha}. \tag{3.50}$$

The evolution of the intensity profile of the dark soliton (3.49) is also shown in figure 2 for the parametric values $\alpha = -1$, $\tau_1^2 = 0.4$, $K_1 = P_1 = 2$, $\Gamma = -4$ and $\lambda = 1$, for (*a*) $\varepsilon = 0.05$ and (*b*) $\varepsilon = 0$.

Next, in order to construct dark two-soliton solutions, we set $g_j = h_j = f_j = 0$ for $j \geqslant 3$ and then proceeding as before we obtain

$$g_1 = h_1 = Z_1 \exp(\xi_1) + Z_2 \exp(\xi_2) \qquad f_1 = \exp(\xi_1) + \exp(\xi_2) \tag{3.51}$$

$$g_2 = h_2 = A_{12} Z_1 Z_2 \exp(\xi_1 + \xi_2) \qquad f_2 = A_{12} \exp(\xi_1 + \xi_2) \tag{3.52}$$

where

$$\xi_j = P_j t - \{[3\varepsilon(K_1^2 - \Gamma) + K_1\lambda]P_j - \varepsilon P_j^3 - P_j(3\varepsilon K_1 + \tfrac{1}{2}\lambda)[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}\}z + \xi_j^{(0)} \tag{3.53}$$

$$Z_j = \frac{-P_j + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}}{P_j + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}} \qquad j = 1, 2 \tag{3.54}$$

and

$$A_{12} = \frac{(P_1 - P_2)^2 + \{[4\alpha_1(\tau_1^2 + \tau_2^2) - P_1^2]^{1/2} - [4\alpha_1(\tau_1^2 + \tau_2^2) - P_2^2]^{1/2}\}^2}{(P_1 + P_2)^2 + \{[4\alpha_1(\tau_1^2 + \tau_2^2) - P_1^2]^{1/2} - [4\alpha_1(\tau_1^2 + \tau_2^2) - P_2^2]^{1/2}\}^2}. \tag{3.55}$$

Here P_j and $\xi_j^{(0)}$ are all real constants. Using (3.37), (3.39) and (3.51)–(3.55) in (3.3), the dark two-solitons can be found explicitly.

In this way by proceeding further, the dark N-soliton solutions can be derived using the following equations in (3.3).

$$g = \tau_1 \exp(\mathrm{i}\zeta_1)\left\{\sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j(\xi_j + \mathrm{i}\theta_j) + \sum_{i<J}^{N} a_{ij}\mu_i\mu_j\right]\right\} \tag{3.56}$$

$$h = \tau_2 \exp(\mathrm{i}\zeta_1)\left\{\sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j(\xi_j + \mathrm{i}\theta_j) + \sum_{i<j}^{N} a_{ij}\mu_i\mu_j\right]\right\} \tag{3.57}$$

$$f = \sum_{\mu=0,1} \exp\left[\sum_{j=1}^{N} \mu_j\xi_j + \sum_{i<j}^{N} a_{ij}\mu_i\mu_j\right] \tag{3.58}$$

where

$$\xi_j = P_j t - \{[3\varepsilon(K_1^2 - \Gamma) + K_1\lambda]P_j - \varepsilon P_j^3 - P_j(3\varepsilon K_1 + \tfrac{1}{2}\lambda)[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}\}z + \xi_j^{(0)} \tag{3.59}$$

$$\exp(\mathrm{i}\theta_j) = \frac{-P_j + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}}{P_j + \mathrm{i}[4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}} \tag{3.60}$$

and

$$\exp(a_{ij}) = \frac{(P_i - P_j)^2 + \{[4\alpha_1(\tau_1^2 + \tau_2^2) - P_i^2]^{1/2} - [4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}\}^2}{(P_i + P_j)^2 + \{[4\alpha_1(\tau_1^2 + \tau_2^2) - P_i^2]^{1/2} - [4\alpha_1(\tau_1^2 + \tau_2^2) - P_j^2]^{1/2}\}^2} \quad \text{for } i, j = 1, 2, \ldots, N. \tag{3.61}$$

4. Discussion

In this paper, considering a generalized set of CHNLS equations, we have found bright and dark N-soliton solutions using the relation between P-analysis and Hirota technique. The model system can govern the dynamics of nonlinear short pulses, which includes femtosecond soliton pulses when fibre loss is excluded. Strictly speaking in the femtosecond regime when fibre loss is substantial the contribution of the self-induced Raman effect becomes important; however, we have not considered this effect in the present work. The nature of soliton solutions reported is in confirmity with the fact that bright solitons occur only if α takes a positive value in order to allow the same sign for the dispersion and cubic nonlinear coefficients as expected in the anomalous GVD region and dark solitons appear only if α takes a negative value in order to allow opposite signs for the dispersion and cubic nonlinear coefficients as expected in the normal GVD region.

The bright one-soliton solution agrees exactly with that obtained from the inverse scattering method [17] and the remaining higher-order bright solitons and all the dark solitons are reported for the first time by taking into account the effects of higher-order terms and the most general linear cross coupling terms systematically. We also noted that the procedure followed here to find higher-order soliton solutions is not really complicated. Further, we expect that the simple form of the reported coupled solitons could be observed experimentally in properly tailored optical fibres. It will also be of use to study whether non-integrable but partially integrable systems of (1.1) can admit special solutions of interest. Work is in progress along these lines.

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